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NOTE

ADDENDUM TO SCHRIJVER'S WORK ON MINIMUM PERMANENTS

IAN M. WANLESS*

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Let Λ_n^k denote the set of $n\times n$ binary matrices which have each row and column sum equal to k. For $2\le k\le n\to\infty$ we show that $(\min_{A\in\Lambda_n^k}\operatorname{per}A)^{1/n}$ is asymptotically equal to $(k-1)^{k-1}k^{2-k}$. This confirms Conjecture 23 in Minc's catalogue of open problems.

Let Δ_n^k denote the set of $n\times n$ matrices of non-negative integers which have each row and column sum equal to k. Let Λ_n^k denote the subset of all binary matrices (matrices of zeroes and ones) in Δ_n^k.

If G is a bipartite multigraph let $\mathcal{B}(G)$ denote the usual 'biadjacency' matrix of G. That is, $\mathcal{B}(G)$ is the matrix with rows and columns respectively corresponding to the vertices in the two parts of G, and with each entry recording how many edges there are between the vertices corresponding to the row and column in which the entry lies. If G is a k-regular bipartite multigraph on $2n$ vertices then $\mathcal{B}(G)\in\Delta_n^k$. Moreover, $\mathcal{B}(G)\in\Lambda_n^k$ iff G is simple.

Schrijver [5] has studied the minimum permanent in Δ_n^k. Our purpose in this note is to use his results to deduce information about the minimum permanent in Λ_n^k. As an aside, it has been conjectured by Minc (Conjecture 24 in [4]) that the minimum permanents in Λ_n^k and Δ_n^k coincide. Our results will be consistent with that conjecture, but do not seem to represent progress towards its resolution.

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* Written while the author was employed by the Department of Computer Science at the Australian National University.

From Schrijver's paper [5] we know that for integers $n \geq k \geq 1$,

$$\min_{A \in \Lambda_n^k} \operatorname{per} A \geq \min_{A \in \Delta_n^k} \operatorname{per} A > \left(\frac{(k-1)^{k-1}}{k^{k-2}} \right)^n. \tag{1}$$

Schrijver [5] also showed that

$$\min_{A \in \Delta_n^k} \operatorname{per} A \leq \frac{k^{2n}}{\binom{kn}{n}} \tag{2}$$

(this result was obtained earlier in [6], albeit with a crucial typographical error). Together with (1) this allowed him to deduce that

$$\lim_{n \to \infty} \left(\min_{A \in \Delta_n^k} \operatorname{per} A \right)^{1/n} = \frac{(k-1)^{k-1}}{k^{k-2}}.$$

Our goal is to show that a similar result holds in Λ_n^k. We state it more generally, to allow for the possibility of k being an arbitrary function of n.

Theorem 1. *Suppose that* $2 \leq k \leq n$ *as* $n \to \infty$. *Then*

$$\left(\min_{A \in \Lambda_n^k} \operatorname{per} A \right)^{1/n} \sim \frac{(k-1)^{k-1}}{k^{k-2}} \sim \left(\min_{A \in \Delta_n^k} \operatorname{per} A \right)^{1/n}. \tag{3}$$

The result for Δ_n^k is immediate from (1) and (2). To obtain the result for Λ_n^k we need to look more closely at Schrijver's method for proving (2). His technique was to build a random k-regular bipartite multigraph M on $2n$ vertices using a method which is known in random graph theory as the *pairing model* (see [7] for a survey of applications of the pairing model and [3] for its specific application to bipartite graphs). He then showed that E, the expected value of $\operatorname{per} \mathcal{B}(M)$, is no more than the right-hand side of (2), from which his result follows.

By construction $\mathcal{B}(M) \in \Delta_n^k$. McKay and Wang [3] showed for $k = o(\sqrt{n})$ that the probability q that $\mathcal{B}(M)$ is in Λ_n^k satisfies

$$\frac{1}{q} = \exp\left(\frac{(k-1)^2}{2} + \frac{(k-1)^2 k}{6n} + o(1) \right). \tag{4}$$

But we know that

$$q \left(\min_{A \in \Lambda_n^k} \operatorname{per} A \right) \leq E \leq \frac{k^{2n}}{\binom{kn}{n}}.$$

It then follows, using (1), that

$$\left(\frac{(k-1)^{k-1}}{k^{k-2}}\right)^n \leq \min_{A\in\Lambda_n^k} \operatorname{per} A \leq \frac{k^{2n}}{q\binom{kn}{n}}. \tag{5}$$

Together with (4) and Stirling's formula this implies (3) for $k=o(\sqrt{n})$.

For larger k, we need only observe that when $k\to\infty$,

$$\frac{(k-1)^{k-1}}{k^{k-2}} = k\left(1-\frac{1}{k}\right)^{k-1} \sim \frac{k}{e} \sim k!^{1/k}.$$

This demonstrates that the lower bound on $(\min_{A\in\Lambda_n^k} \operatorname{per} A)^{1/n}$ from (1) and the upper bound on $(\max_{A\in\Lambda_n^k} \operatorname{per} A)^{1/n}$ from Brègman's theorem [1] agree in this case. Theorem 1 now follows, and from it we deduce the truth of Conjecture 23 in [4]. For further discussion of this conjecture and a progress report on all of Minc's open problems, see [2].

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Ian M. Wanless

School of Mathematical Sciences
Monash University
Vic 3800
Australia
ian.wanless@sci.monash.edu.au